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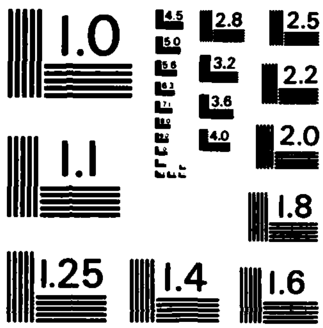

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Research Report CCS 471

HOMOMONOTONE GAMES AND MONOCORE SOLUTIONS

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Homomonotone Games and Solutions

by

A. Charnes, S. Duffuaa, B. Golany

Abstract

A new solution concept, the Monocore, for superadditive n-person cooperative games is introduced. Based on the notion of strong-superadditivity or "homomonotonicity", it transforms any superadditive game into a non-empty core game and then finds a unique imputation in the core of the transformed game. The Monocore is applied to games in characteristic function form and in homomollifier form and a way of comparison is suggested. A general class of Monocore solutions is described and related to some aspects of Information Theory.

Key Words

Monocore

Homomonotonicity

Homomollifier

Core

Minimum Discrimination Information

1. Introduction

In a recent paper, [1], we considered deficiencies and advantages connected with the solution concept of the core for n-person superadditive games. There, we suggested the "homocore" as a new solution concept, based on the homomollifier notion (see also [2]) and the core idea of stability. The homocore is the product of a process which generates an empty core extension from any superadditive really essential game. In the extension the value of the grand coalition is raised, so that all the inequalities in the core system can be satisfied. The homocore is obtained as a unique imputation based on average values for each player from coalitions in which he participates and at a level which is last to be satisfied in the above system.

Here we approach the problem of defining a core-like solution from a somewhat different direction. We start by defining a per-person superadditivity property which we call "homomonotonicity." The usual superadditivity property guarantees to any two disjoint coalitions S and T that they can acquire at least as much by forming the larger coalition $S \cup T$ as by remaining separated. The usual superadditivity property might still allow a situation in which the average payoff to players in S or T is reduced in the joint coalition. This cannot happen in a "homomonotone" game in which the much stronger property of per-person superadditivity holds.

Some interesting consequences of homomonotonicity are that it implies: (i) usual superadditivity, (ii) a non-empty core. Homomonotonicity thus is a desirable outcome of a game mapping which would put us in an opposite position to that attained in [1], where we had a transformation which mapped any game into an empty core homomollifier. Here, in section 3, we define a transformation which will map any game into a homomonotone game.

Based on the fact that the transformed game has a non-empty core, we define a process which yields a new solution notion for an n-person game which we name "Monocore." This solution is a unique imputation obtained by considering only the two last levels of inequalities in the system which determines the core. This is possible since we prove that it is always the $(n-1)^{th}$ level in the transformed game which is critical in determining the core. We suggest two ways to reach the Monocore solution--via construction or through solving a quadratic programming problem. We also compare the monocore over the original game (N,v) to the monocore derived over the homomollifier (N,w) of the game. In section 5, we define a generalized monocore solution starting with any of a class of specifications of the personal contribution of each player to the grand coalition.

2. Properties of Homomonotonicity

Let (N,v) be a game in characteristic function form where $N = \{1,2,\dots,n\}$ is the set of players and v is a non-negative real valued function defined on all the subsets of N with $v(\emptyset) = 0$, $x = (x_1, x_2, \dots, x_n)$ be a payoff vector, S any subset of N with s the cardinality of S. By x(S) we mean $\sum_{i \in S} x_i$. The complement of the game, denoted by $\bar{v}$, is defined by $\bar{v}(S) = v(N) - v(N - S)$ and the homomollifier W is given by: $w(S) = \frac{s}{n}\bar{v}(S) + \frac{n-s}{n} v(S)$. A game is called "superadditive" if for all S, $T \subset N$ with $S \cap T = \emptyset$, $v(S) + v(T) \leq (S \cup T)$.

Definition I: A game is "homomonotone" if for every S, T such that $S \subseteq T$, $\frac{v(S)}{s} \leq \frac{v(T)}{t}$.

If each player in a coalition S is sure to receive the average of the coalition worth v(S), it is clearly to his advantage to join as large a coalition as possible in a homomonotone game. Thus with respect to expectation of equal distribution of coalitional worth the grand coalition is the coalition most likely to form in a homomonotone game.

Theorem 2.1: A homomonotone game is superadditive.

Proof: consider any, $s_1, s_2, \subseteq N$ with $S_1 \cap S_2 = \phi$. Let $T = S_1 \cup S_2$. Then by homomonotonicity we have

$$\frac{v(S_1)}{s_1} \leq \frac{v(T)}{t}, \quad \frac{v(S_2)}{s_2} \leq \frac{v(T)}{t}$$

or

$$v(S_1) \leq \frac{s_1}{t} v(T), \quad v(S_2) \leq \frac{s_2}{t} v(T)$$

Adding the inequalities, $v(S_1) + v(S_2) \leq \frac{s_1+s_2}{t} v(T) = v(T)$. Hence the game is superadditive.

Q.E.D.

Theorem 2.2: Every subgame of a homomonotone game is homomonotone.

Proof: Consider the subgame (T,w) defined by the coalition $T \subseteq N$ and all subsets of T with characteristic function values for T and its subsets precisely those of the original game.

Consider any pair of subsets $S_1 \subseteq S_2 \subseteq T$.

Then $$\frac{w(S_1)}{s_1} = \frac{v(S_1)}{s_1} \le \frac{v(S_2)}{s_2} = \frac{w(S_2)}{s_2}$$

So (T, w) is a homomonotone game.

Q.E.D.

Theorem 2.3:

A non-negative homomonotone game has a non-empty core.

Proof: By the homomonotonicity property,

$$\frac{v(S)}{s} \le \frac{v(N)}{n}, \quad \forall \; S \subseteq N$$

If $v(N) = 0$ then so is $v(S)$, $\forall \; S \subseteq N$. Thus $x_i = 0$, $i = 1,\dots n$ satisfies the core properties. If some $v(S) > 0$, then by the homonotonicity $v(N) > 0$.

Since $v(i) \le \frac{v(N)}{n}$, $\forall i \in N$, choose $x_i = \frac{1}{n} v(N)$

Then $x(S) \triangleq \sum_{i \in S} x_i = \frac{s}{N} v(N) \ge v(S)$ by homomonotonicity for all $S \subseteq N$.

Also $x(N) = \sum_{i=1}^{n} x_i = n \cdot \frac{1}{n} v(N) = v(N)$.

Hence $x^T \triangleq (\frac{1}{n} v(N), \dots, \frac{1}{n} v(N))$ is in the core.

Q.E.D.

Remark 2.1: Although it might be tempting to think that homomonotonicity implies convexity of the game (since any convex game has a core), this is not true, e.g. the following is a homomonotone game which is not convex:

$$v(1) = 1, \; v(2) = \frac{1}{3}, \; v(3) = 1$$
$$v(12) = 3, \; v(23) = 4, \; v(13) = 4$$
$$v(123) = 6$$

The game is homomonotone since

$$v_1 \le \frac{v(12)}{2}, \frac{v(13)}{2}, \frac{v(123)}{3}; \quad v_2 \le \frac{v(12)}{2}, \frac{v(23)}{2}, \frac{v(123)}{3};$$

$$v_3 \le \frac{v(13)}{2}, \frac{v(23)}{2}, \frac{v(123)}{3}; \quad \frac{v(12)}{2} \le \frac{v(123)}{3}, \quad \frac{v(13)}{2} \le \frac{v(123)}{3},$$

$$\frac{v(23)}{2} \le \frac{v(123)}{3}$$

But it is not convex: $v(12) + v(23) = 3 + 4 = 7 > v(123) + v(2) = 6\frac{1}{3}$

contradicting $v(S_1) + v(S_2) \le v(S_1 \cup S_2) + v(S_1 \cap S_2)$, $\forall\, S_1, S_2 \subseteq N$,

the convexity property.

Remark 2.2: It might be thought that with the strong increase in sizes of $v(S)$ with the homomonotonicity property, any homomonotone game would be essential. However, this is also not true, e.g. the game $v(S) = s$ defines a homomonotone inessential game.

So far we have discussed some of the properties of a homomonotone game and now we turn to a process, "homo-transformation," which maps any game into a homomonotone game.

Definition 2: The homo-transform (N,v') of (N,v) is given by

$$v'(S) = \frac{s}{n} v(S), \quad \forall\, S \subseteq N.$$

Theorem 2.4: The homo-transform (N,v') of a superadditive game (N,v) is homomonotone.

Proof: Since (N,v) is a non-negative superadditive game, $v(S) \le v(T)$, $\forall\, S \subseteq T \subseteq N$.

Hence $\frac{s}{n}\,\frac{v(S)}{s} \le \frac{t}{n}\,\frac{v(T)}{t}$.

By definition 2 we get $\frac{v'(S)}{s} \le \frac{v'(T)}{t}$, $\forall\, S \subseteq T \subseteq N$

i.e. (N,v') is homomonotone,

Q.E.D.

3. The Monocore Solution

The transformation in 2.4 guarantees the existence of the core, but still we are left with the non-uniqueness of the core. The following construction will always select a unique member of the core set.

The system of inequalities which determine the core (of the transformed game v') is

$$
(1)\qquad
\begin{array}{lll}
\left.\begin{array}{l}
x_1 \geq v'_1 \\
x_2 \geq v'_2 \\
\quad\vdots \\
x_n \geq v'_n
\end{array}\right\} & \text{1st level} \\[2ex]
\left.\begin{array}{l}
x_1+x_2 \geq v'_{1,2} \\
\quad\vdots \\
x_{n-1}+x_n \geq v'_{n-1,n}
\end{array}\right\} & \text{2nd level} \\
\quad\vdots & \\
\left.\begin{array}{l}
x_1+x_2+\dots+x_{n-2} \geq v'_{1,2,3,\dots,n-2} \\
\quad\vdots \\
x_3+x_4+\dots+x_{n-1}+x_n \geq v'_{3,4,\dots,n}
\end{array}\right\} & n\text{-}2^{th}\ \text{level} \\[2ex]
\left.\begin{array}{l}
x_1+x_2+\dots+x_{n-1} \geq v'_{1,2,3,\dots,n-1} \\
\quad\vdots \\
x_2+x_3+\dots+x_{n-1}+x_n \geq v'_{2,3,\dots,n}
\end{array}\right\} & n\text{-}1^{th}\ \text{level} \\[2ex]
\left.\begin{array}{l}
x_1+x_2+\dots+x_n = v'(N)
\end{array}\right\} & n^{th}\ \text{level.}
\end{array}
$$

From theorem 2.3 we know that this game has a non-empty core, i.e., this system of inequalities is consistent.

The next lemma shows that in our case, we need to consider just one level out of the n levels above.

<u>Lemma</u>: The critical level for (N,v') is the $(n-1)^{th}$ level.

<u>Proof</u>: Let T^i be the n-1 player set omitting player i and S^i_j be the j^{th} set of k members omitting player i.

Then

$$T^i \supseteq S^i_j \quad , \quad j = 1,\dots,\binom{n-1}{k} ,$$

Note that in this notation each set S of k members has n-1 designations of form S^i_j as may be seen from the incidence matrix of players in k member sets.

By homomonotonicity,

$$\frac{v(T^i)}{n-1} \geq \frac{v(S^i_j)}{k} \quad , \quad j = 1,\dots,\binom{n-1}{k} \; , \; \forall\, i$$

Hence $$\frac{\binom{n-1}{k}}{n-1}\, v(T^i) \geq \frac{1}{k} \sum_j v(S^i_j)$$

and $$\frac{\binom{n-1}{k}}{n-1} \sum_i v(T^i) \geq \frac{1}{k} \sum_i \sum_j v(S^i_j) = \frac{n-k}{k} \sum_{s=k} v(S)$$

Thus $$\frac{1}{n-1} \sum_{t=n-1} v(T) \geq \frac{1}{k} \frac{n-k}{\binom{n-1}{k}} \sum_{s=k} v(S)$$

and $$\frac{1}{n-1} \left\{ \frac{1}{n} \sum_{t=n-1} v(T) \right\} \geq \frac{n-k}{n\binom{n-1}{k}} \frac{1}{k} \sum_{s=k} v(S) = \frac{n-k}{(n-k)\binom{n}{k}} \frac{1}{k} \sum_{s=k} v(S) ,$$

since $$n\binom{n-1}{k} = (n-k)\binom{n}{k} .$$

Thus,
$$\frac{1}{n-1} \left\{ \frac{1}{n} \sum_{t=n-1} v(T) \right\} \geq \frac{1}{k} \frac{1}{\binom{n}{k}} \sum_{s=k} v(S) .$$

Q.E.D.

Hence, the maximum of $\frac{1}{\binom{n-1}{s-1}} \sum_{s=k} v'(S)$ occurs when $k = n-1$.

Q.E.D.

Now suppose we change all the inequalities in the $(n-1)^{th}$ level to equalities and solve the sub-system which consists of the last n+1 equations in (1).

We know by theorem 2.3 and the previous lemma that our system has many inputations in the core. We attain the monocore by the following process:

(i) Let $\tilde{x}(i) = v'(N) - v'(N-i) = v(N) - \frac{n-1}{n} v(N-i)$.

Then

(ii) $$\sum_{i=1}^{n} \tilde{x}(i) = nV(N) - \frac{n-1}{n} \sum_{i=1}^{n} v(N-i) = nV(N) - (n-1)A\ ,$$

where $A = \frac{1}{n} \sum_{i=1}^{n} v(N-i)$ is the average value for characteri functions in the $(n-1)^{th}$ level.

It is clear that both $\tilde{x}(i) > 0$ and $\sum_{i=1}^{n} \tilde{x}(i) \geq v(N)$ hold.

(iii) Since we need $x(N) = v(N)$ for x to be an imputation, we should scale $\tilde{x}(i)$ "downward" by $v(N)/(nv(N) - (n-1)A)$.

Hence, we define the monocore as:

$$x(i) = \boxed{\frac{\left[v(N) - \frac{n-1}{n} v(N-i)\right] v(N)}{nv(N) - (n-1)A}} \tag{2}$$

Theorem 3.1: The monocore of any superadditive (N,v) game always exists, is unique, and is an imputation in the core of the transformed game (N,v').

Proof: The existence and uniqueness properties are trivially given by the construction of x(i) above.

For x to be an imputation in the core, we need to prove that $x(S) \geq v'(S)$, $\forall\ S \subset N$ and $x(N) = v'(N)$. By the lemma above, we need to consider only $|S| = n-1$. Also $x(N) = v'(N)$ by construction. So we have only to prove that:

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} x(j) \geq v'(N-i)\ , \quad \forall \quad i=1,\ldots,n$$

Add $x(i)$ to both sides:

$$v(N) \geq \frac{n-1}{n} v(N-i) + \left[v(N) - \frac{n-1}{n} v(N-i)\right]\frac{v(N)}{v''(N)}\ ,$$

where $v''(N)$ is the denominator in (2).

By the construction, $\frac{v(N)}{v''(N)} < 1$, so

$$\frac{n-1}{n} v(N-i) + \left[v(N) - \frac{n-1}{n} v(N-i)\right]\frac{v(N)}{v''(N)} < v(N).$$

Q.E.D.

Next, we notice that $x(i)$ can be written as:

(3) $$x(i) = \rho_i v(N) \quad \text{where} \quad \rho_i = \frac{v(N) - \frac{n-1}{n} v(N-i)}{nv(N) - (n-1)A}$$

and $$\sum_{i=1}^{n} \rho_i = 1.$$

Recall theorem 2.3 where we used the imputation $x(i) = \frac{1}{n} v(N)$. We are now back in a situation where $x(i)$ is expressed as a proportion of $v(N)$.

<u>Theorem 3.2</u>: The proportion ρ_i is greater (smaller) than the "uniform" proportion $\rho_i = \frac{1}{n}$, $\forall$ i, only if $v(N-i) \leq A$ $(v(N-i) \geq A)$.

<u>Proof</u>: Suppose $v(N-i) \leq A$. Then:

$$\rho_i = \frac{v(N) - \frac{n-1}{n} v(N-i)}{nv(N) - (n-1)A} \geq \frac{v(N) - \frac{n-1}{n} A}{nv(N) - (n-1)A} \geq \frac{1}{n} \quad \text{as asserted.}$$

Similarly, the $v(N-i) \geq A$ proposition is demonstrated.

Q.E.D.

Our purpose in introducing the proportions ρ_i in (3) becomes clearer by the next theorem. Here we wish to interpret the monocore solution concept as an optimal solution to a certain extremal principle.

Theorem 3.3: For ρ_i given as in (3), the monocore is the solution to the following quadratic programming problem.

$$\text{Min} \qquad \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\rho_i}x_i^2$$

$$\text{s.t.} \qquad \sum_{i=1}^{n}x_i = v(N).$$

Proof: Differentiating the Lagrangean, z, and setting it equal to zero,

$$\frac{\partial z}{\partial x_i} = \frac{1}{\rho_i}x_i - \lambda = 0. \text{ Hence } x_i = \rho_i\lambda$$

$$\frac{\partial z}{\partial \lambda} = \sum_i x_i - v(N) = 0$$

Hence $\quad \lambda = v(N)/\sum_i \rho_i$

which implies $\quad x_i = \dfrac{v(N)}{\sum_j \rho_j}\rho_i = \rho_i v(N).$

Q.E.D.

To see what the monocore is for games without a core, with a unique core, and with many imputations in the core, consider the following examples.

Example 3.1: An empty core game

$$v(1) = v(2) = v(3) = 0 \text{ , } v(12) = v(123) = 1 \text{ , } v(13) = \frac{5}{6} \text{ , } v(23) = \frac{4}{6}.$$

Now $A = \frac{1}{3}\sum_{|S|=2}v(S) = \frac{5}{6}$. So $nv(N) - (n-1)A = \frac{8}{6}$.

Hence the monocore is: $x(1) = \left[1 - \frac{2}{3} \cdot \frac{4}{6}\right] \frac{1}{8/6} = 5/12$

$$x(2) = \left[1 - \frac{2}{3} \cdot \frac{5}{6}\right] \frac{1}{8/6} = 4/12$$

$$x(3) = \left[1 - \frac{2}{3} \cdot 1\right] \frac{1}{8/6} = 3/12$$

Example 3.2: A game with unique core (BL^2)

$$v(1) = v(2) = v(3) = v(23) = 0 \; ; \; v(12) = v(13) = v(123) = 1.$$

$A = \frac{2}{3}$ and $nv(N) - (n-1)A = \frac{5}{3}$. The monocore turns out to be

$$x(1) = \frac{3}{5}, \; x(2) = \frac{1}{5}, \; x(3) = \frac{1}{5}.$$

This happens to be the "disruption" solution in [2], but this solution property does not hold in general.

Example 3.3: A game with an infinite number of imputations in the core

$$v(1) = v(2) = v(3) = 0 \; ; \; v(12) = v(123) = 1 \; ; \; v(13) = \frac{1}{2}, \; v(23) = 0.$$

$A = \frac{1}{2}$ and $nv(N) - (n-1)A = 2$. The monocore is

$$x(1) = \frac{1}{2}, \; x(2) = \frac{1}{3}, \; x(3) = \frac{1}{6}.$$

4. The Monocore of the Homomollifier

We suggested in [1] that the homomollifier for each game represents the conclusion of an implicit bargaining process. We now apply the monocore solution concept to the homomollifier. The steps for doing this are as follows:

(1) Find the homomollifier $w(S)$ for the given game.

(2) Apply the homo-transform defined in 2.4 to get $w'(S)$.

(3) Use the construction or the extremal principle to get the monocore solution to the game $w(S)$.

The next theorem will show the relation between the monocore of the homomollifier and the monocore of the original game (N,v).

First, we define a measure of inequality in the distribution of payoffs as follows:

$$(4) \qquad M = \sum_{i=1}^{n}\left[\frac{v(N)}{n} - x(i)\right]^2$$

Clearly, for the "uniform" imputation used in theorem 2.3, this measure is zero. More generally, for any solution concept this measure describes the amount of dispersion in payoffs of the group players.

Theorem 4.1: The dispersion measure M is always smaller for the monocore of the homomollifier game (N,w) than for the monocore of the original game (N,v).

Proof: W.l.o.g. we will prove the theorem for games in 0-1 normalization, since any superadditive game in characteristic function form is strategically equivalent to such a game. Let M_v, M_w be the respective measures for (N,v) and (N,w). Then

$$M_v = \sum_{i=1}^{n}\left[\frac{v(N)}{n} - x(i)\right]^2 = \sum\left[\frac{v(N)}{n} - \frac{v(N)}{n}\,\frac{v(N)-\frac{n-1}{n}\,v(N-i)}{v(N)-\frac{n-1}{n}\,A}\right]^2 =$$

$$= \frac{(n-1)^2v^2(N)}{n^4}\sum_{i=1}^{n}\left[\frac{v(N-i)\ -A}{v(N)\ -\frac{n-1}{n}\,A}\right]^2$$

Similarly $\quad M_w = \dfrac{(n-1)^2v^2(N)}{n^4}\displaystyle\sum_{i=1}\left[\frac{w(N-i)\ -\bar{A}}{v(N)\ -\frac{n-1}{n}\,\bar{A}}\right]^2$, where $\bar{A} = \frac{1}{n}\displaystyle\sum_{i=1}^{n}w(N-i)$.

Now $w(N-i) = \frac{n-1}{n}(v(N)-v(i)) + \frac{1}{n}v(N-i) = \frac{1}{n}\left[v(N-i) + v(i) - v(N)\right] + v(N)$

and $\bar{A} = \frac{1}{n}\sum_{i=1}^{n} w(N-i) = \frac{1}{n}\left[A + a - v(N)\right] + v(N)$ where $a = \frac{1}{n}\sum_{i=1}^{n} v(i)$.

Hence,

$$[w(N-i) - \bar{A}]^2 = \frac{1}{n^2}\left[(v(N-i) - A) + (v(i) - a)\right]^2.$$

Further, the denominator of M_w is given by

$$v(N) - \frac{n-1}{n}\bar{A} = \frac{1}{n}\left[v(N) - \frac{n-1}{n}[A + a - v(N)]\right] = \frac{1}{n^2}[(2n-1)v(N) - (n-1)(A+a)].$$

In the 0-1 normalization, $v(i) = 0$, $\forall\, i$, hence $a = 0$. Thus we have for the i^{th} term in M_w

$$\left[\frac{v(N-i) - A}{(2n-1)v(N) - (n-1)A}\right]^2$$

To complete the proof we only need to show that

$(2n-1)v(N) - (n-1)A \geq v(N) - \frac{n-1}{n}A$, which is easy to see by collecting terms.

Q.E.D.

Let us now compare the monocore solutions for the homomollifiers of our examples 3.1 to 3.3:

<u>Example 4.1</u>: The game (N,v) is as in 3.1. The homomollifier is

$w(12) = 1$, $w(13) = \frac{17}{18}$, $w(23) = \frac{16}{18}$, $w(123) = 1$. Then $\bar{A} = \frac{17}{18}$ and the

monocore is: $$x(1) = \left[\frac{1 - \frac{2}{3}\cdot\frac{16}{18}}{1 - \frac{2}{3}\cdot\frac{17}{18}}\right]\frac{1}{3} = \frac{11}{30}$$

$$x(2) = \left[\frac{1 - \frac{2}{3}\cdot\frac{17}{18}}{1 - \frac{2}{3}\cdot\frac{17}{18}}\right]\frac{1}{3} = \frac{10}{30}$$

$$x(3) = \left[\frac{1 - \frac{2}{3}\cdot\frac{18}{18}}{1 - \frac{2}{3}\cdot\frac{17}{18}}\right] \frac{1}{3} = \frac{9}{30}$$

Here $M_w = \frac{2}{900} = 0.0022$ while $M_v = \frac{2}{144} = 0.0138$. Clearly, the monocore over the homomollifier has a much more uniform distribution of payoffs.

Example 4.2: The game (N,v) is as in 3.2. The homomollifier is

$w(12) = w(13) = w(123) = 1$, $w(23) = \frac{2}{3}$. Then $\bar{A} = \frac{8}{9}$, and the monocore is:

$$x(1) = \left[\frac{1 - \frac{2}{3}\cdot\frac{2}{3}}{1 - \frac{2}{3}\cdot\frac{8}{9}}\right] \frac{1}{3} = \frac{15}{33}$$

$$x(2) = \left[\frac{1 - \frac{2}{3}\cdot 1}{1 - \frac{2}{3}\cdot\frac{8}{9}}\right] \frac{1}{3} = \frac{9}{33}$$

$$x(3) = \frac{9}{33}$$

Here $M_w = 0.0220$ while $M_v = 0.1066$.

Example 4.3: The game is that of 3.3. The homomollifier is $w(12) = 1$, $w(13) = \frac{5}{6}$, $w(23) = \frac{4}{6}$, $w(123) = 1$. Then $\bar{A} = \frac{5}{6}$ and the monocore is:

$$x(1) = \left[\frac{1 - \frac{2}{3}\cdot\frac{4}{6}}{1 - \frac{2}{3}\cdot\frac{5}{6}}\right] \frac{1}{3} = \frac{10}{24}$$

$$x(2) = \left[\frac{1 - \frac{2}{3}\cdot\frac{5}{6}}{1 - \frac{2}{3}\cdot\frac{5}{6}}\right] \frac{1}{3} = \frac{8}{24}$$

$$x(3) = \left[\frac{1 - \frac{2}{3}\cdot 1}{1 - \frac{2}{3}\cdot\frac{5}{6}}\right] \frac{1}{3} = \frac{6}{24}$$

Again $M_w = 0.0138 < M_v = 0.055$.

5. A Class of Monocore Solutions

In this section, we define another simple extremal principle which yields as a special case the monocore solution of the previous section. We use the well-known notion of Minimum Discrimination Information (see e.g. [3], [4]).

Let z_i be any strictly positive measure of the i^{th} player's contribution to the grand coalition. A "generalized" monocore is then the solution to the following non-linear (goal) programming problem with the z_i as goals:

$$\text{(5)} \quad \begin{aligned} &\text{Min} \quad c = \sum_{i=1}^{n} x_i \ln \frac{x_i}{z_i} \\ &\text{s.t.} \\ &\qquad \sum_{i=1}^{n} x_i = v(N) \ . \end{aligned}$$

Still more generally, we may impose additional constraints.

Lemma: The explicit solution to (5) is $x_i = \dfrac{z_i}{\sum_{i=1}^{n} z_i} v(N)$.

Proof: Differentiating the Lagrangean, $\hat{c}$, of (5) and setting it to zero,

$$\text{(6)} \qquad \frac{\partial \hat{c}}{\partial x_i} = \ln \frac{x_i}{z_i} + x_i \frac{z_i}{x_i} \frac{1}{z_i} + \lambda = 0$$

$$\text{(7)} \qquad \frac{\partial \hat{c}}{\partial \lambda} = \sum_{i=1}^{n} x_i - v(N) = 0$$

From (6), we get $x_i = z_i e^{-(1+\lambda)}$ and $\sum_{i=1}^{n} x_i = e^{-(1+\lambda)} \sum_{i=1}^{n} z_i$.

Using (7), we conclude $x_i = \dfrac{z_i}{\Sigma z_i} v(N)$.

Q.E.D.

Remark 5.1: Notice that if we choose $z_i = v(N) - \frac{n-1}{n} v(N-i)$ as a measure of the i^{th} player's contribution to the grand coalition, the generalized monocore is the monocore of our construction in §3.

Remark 5.2: If instead we choose

$$z_i = \sum_{\substack{|S|=k \\ i \in S}} w(S) - \frac{k-1}{n-1} \sum_{|S|=k} w(S) \text{ , where k is such that}$$

$$\max_{|S|<n} \frac{1}{\binom{n-1}{|S|-1}} \sum_{|S|} w(S) = \frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} w(S) \text{ ,}$$

the generalized monocore is precisely the homocore of [1].

6. Conclusion

A new unique core-like concept of solution for n-person games is defined based on a new property called the homomonotonicity. This solution is called the monocore and it is based on a simple game mapping which will map any n-person superadditive game into a homomonotone game. Every homomonotone game has a non-empty core.

This solution concept conforms to the idea that the homomollifier represents an implicit bargaining process. Also this concept of solution is characterized by an extremal principle from which a generalized monocore solution is proposed.

Further research will study different transformations of games to attain homomonotone games and their effect on this concept of solution.

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A new solution concept, the Monocore, for superadditive n-person cooperative games is introduced. Based on the notion of strong-superadditivity or "homomonotonicity", it transforms any superadditive game into a non-empty core game and then finds a unique imputation in the core of the transformed game. The Monocore is applied to games in characteristic function form and in homomollifier form and a way of comparison is suggested. A general class of Monocore solutions is described and related to some aspects of Information Theory.